# Response function of coarsening systems

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**Abstract.** The response function of domain growth processes, and in particular the violation of the fluctuation-dissipation theorem, are studied both analytically and numerically. In the asymptotic limit of large times, the fluctuation-dissipation ratio $X$, which quantifies this violation, tends to one if $C > m^2$ and to zero if $C < m^2$, corresponding to the fast ("bulk") and slow ("domain-wall") responses, respectively. In this paper, we focus on the pre-asymptotic behavior of the domain-wall response. This response is shown to scale with the typical domain length $L(t)$ as $1/L(t)$ for dimension $d > 2$, and as $\ln(L(t))/L(t)$ for $d = 2$. Numerical results confirming this analysis are presented.

**PACS.** 05.70.Ln Nonequilibrium and irreversible thermodynamics – 75.40.Gb Gb Dynamic properties (dynamic susceptibility, spin waves, spin diffusion, dynamic scaling, etc.) – 75.40.Mg Numerical simulation studies

## 1 Introduction

Domain growth systems are the paradigm of systems that do not reach equilibrium. Hence, it has been a recurring theme in the field of spin and structural glasses to think these systems as displaying some form of coarsening [1,2]. In such non-equilibrium systems, time translation invariance does not hold, and all time dependent correlation functions and response functions depend on two times (the time origin corresponding generally to the time at which the system has been quenched into the non-equilibrium state). In domain growth phenomena, an autocorrelation function $C(t, t')$ of the form $C(L(t)/L(t'))$ is usually interpreted as arising from structures whose size grows as $L(t)$. A similar functional form, however, is also found to describe the out of equilibrium dynamics of mean-field models of glasses [3], although there is by construction no length $L$ in such models. In fact, the difference between both kinds of models only becomes manifest when one also considers the *response* functions associated to the correlation functions.

Generally speaking, the fluctuation-dissipation theorem (FDT), which for equilibrium systems relates the response functions to the correlation functions, does not hold in systems that are out of equilibrium. The deviations from FDT are conveniently described by introducing the Fluctuation Dissipation Ratio (FDR) $X(t, t')$ defined through $TR(t, t') = X(t, t')\frac{\partial C}{\partial t'}(t, t')$, where $R$ is a response function and $C$ the associated correlation function. In mean field models of spin glasses, the behavior of this FDR has been well es-

tablished, at least in the asymptotic limit of large times. One has $X \sim 1$ for "fast" processes ($t \sim t'$, large $C$), but a value $0 < X < 1$ for the "slow" processes corresponding to well-separated times [3,4]. This observation prompted several workers [5,6] to calculate the large-time response of pure (non-disordered) coarsening models, in order to quantify the similarities and differences with mean field models of glasses. In order to make a comparison with the glassy case, one computes the staggered response to a spatially random field, to make the perturbation uncorrelated with the equilibrium pure states (as is the case, for example of a uniform field for a spin glass).

The result is that $X \sim 0$ for all but the smallest time differences. In other words, the long-time memory of coarsening systems tends to vanish, unlike in mean-field glass models where it does not since $X > 0$ even at long times. As far as we know, all systems in which two (or only few) phases separate have $X = 0$ at long times, and this has been proven under certain assumptions [7]. Physically, this feature can be understood from the fact that for long times the response will be dominated by the bulk response of the domains that form during the coarsening process. The response at time $t$ of a spin to a field applied at time $t'$ will be nonzero only if the spin is not swept by a domain wall between $t$ and $t'$. Other types of response involve the domain walls themselves, whose density decreases with time, and therefore vanish in the limit of large times.

From the experimental point of view, aging experiments [8] show that glasses such as spin glasses or molecular glasses *do* have long term memory. The asymptotic nature of experimental results is however, always

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questionable. This is even more the case in numerical studies, which for a number of models (spin glasses, structural glasses, kinetic models and polymers in disordered media [9]) have obtained results in qualitative agreement with mean-field theory ($X \neq 0$). It is therefore a relevant question to study the deviations from FDT in the pre-asymptotic limit. An understanding of this pre-asymptotic behavior should allow to distinguish between true long term memory and a slow approach to a vanishing $X$.

In this paper we present such a study for the ferromagnetic coarsening or phase-separation of pure (non-disordered) systems after a quench at time $t = 0$ from a homogeneous phase ($T = \infty$) into a two-phase region ($T < T_c$), with and without local conservation of the order parameter. In the thermodynamic limit, the equilibrium state, where the two phases are completely separated is never achieved. We confirm the previous results for the absence of long term memory in the response function [5,6], and then study the scaling of this response *in the pre-asymptotic regime* (large but finite times). It turns out that the model-dependence enters only through the form of the growth law $L(t)$.

We present the systems and the dynamical quantities under study in Section 2. The numerical simulation is described in the Section 3, while the analytic study is presented in Section 4.

## 2 Models and definitions

The systems considered here will be described by a coarse-grained formulation, with a scalar order parameter $\phi(\mathbf{r}, t)$ and a Ginzburg-Landau free energy functional

$$F[\phi] = \int \mathrm{d}^d r \left[ \frac{1}{2} |\boldsymbol{\nabla}\phi|^2 + \frac{1}{4}\phi^4 - \frac{1}{2}\phi^2 - h\phi \right], \quad (1)$$

where $h(\mathbf{r}, t)$ is the field conjugated to $\phi(\mathbf{r}, t)$. Experimental situations under consideration are for example the coarsening in a ferromagnet, or spinodal decomposition in a binary alloy.

Domain growth processes have been much studied since the early works of Lifshitz, Slyosov and Wagner. Reference [10] is a very complete review on the topic.

If the order parameter is not locally conserved, we have the time-dependent Ginzburg-Landau equation

$$\frac{\partial \phi}{\partial t} = -\frac{\delta F}{\delta \phi} + \eta, \quad (2)$$

where $\eta$ is a Gaussian Markovian noise term satisfying $\langle \eta(\mathbf{r}, t) \rangle = 0$ and $\langle \eta(\mathbf{r}, t)\eta(\mathbf{r}', t') \rangle = 2T\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$, and $T$ is the temperature. When the order parameter is conserved, the evolution is given by the Cahn-Hilliard equation

$$\frac{\partial \phi}{\partial t} = \nabla^2 \left( \frac{\delta F}{\delta \phi} \right) + \eta. \quad (3)$$

In that case, the thermal noise is characterized by the two moments of the Gaussian distribution $\langle \eta(\mathbf{r}, t) \rangle = 0$, and $\langle \eta(\mathbf{r}, t)\eta(\mathbf{r}', t') \rangle = -2T\delta(t - t')\nabla^2\delta(\mathbf{r} - \mathbf{r}')$.

Interesting dynamical quantities in the study of the out of equilibrium properties are the autocorrelation function defined by

$$C(t, t_w) \equiv \frac{1}{V} \int \mathrm{d}^d r \, \langle \phi(\mathbf{r}, t)\phi(\mathbf{r}, t_w) \rangle, \quad (4)$$

and the associated response function $R(t, t_w) \equiv \langle \delta\phi(t)/\delta h(t_w) \rangle$. At equilibrium, these two quantities depend on time difference $\tau \equiv t - t_w$ only, and are related by the usual fluctuation dissipation theorem

$$R(\tau) = -\frac{1}{T} \frac{\partial C(\tau)}{\partial \tau}. \quad (5)$$

Out of equilibrium we write [4]:

$$R(t, t_w) = \frac{X(t, t_w)}{T} \frac{\partial C(t, t_w)}{\partial t_w}, \quad (6)$$

which defines $X(t, t_w)$ as the fluctuation dissipation ratio.

The strategy for our study is now standard. The quench of the system takes place at $t = 0$. In order to compute the correlation $C(t, t_w)$ we record the configurations of the system evolving at zero external field, $h = 0$ for times $t > t_w$. The operation is repeated on several samples in order to improve the statistics.

The integral of the linear response function $M(t, t_w) \equiv \int_{t_w}^t \mathrm{d}s \, R(t, s)$ is computed by letting the system evolve under the influence of a small field switched on at $t_w$, and recording the magnetization at time $t$. The field is random in space and stationary [6]. It is drawn from a Gaussian distribution with first moment $\overline{h(\mathbf{r})} = 0$ and second $\overline{h(\mathbf{r})h(\mathbf{r}')} = h_0{}^2\delta(\mathbf{r} - \mathbf{r}')$, respectively. In the language of magnetic systems, the integrated response function is thus the staggered magnetization

$$M(t, t_w) = \frac{1}{h_0{}^2 V} \int \mathrm{d}^d r \, \overline{\langle h(\mathbf{r})\phi(\mathbf{r}, t) \rangle}. \quad (7)$$

An important property of the FDR has to be emphasized for the present discussion. This property, found analytically within mean-field models and verified numerically in various glassy systems [4,9], is that in the asymptotic regime of $t, t_w \to \infty$, $X(t, t_w)$ depends on the times only through a *non singular* function of correlation function $C(t, t_w)$, that is $X(t, t_w) \equiv x(C(t, t_w))$. When this property holds, the generalized FDT (6) gives the following relation between $M(t, t_w)$ and $C(t, t_w)$:

$$M(t, t_w) = \frac{1}{T} \int_{C(t, t_w)}^{C(t, t)} \mathrm{d}C \, x(C). \quad (8)$$

In equilibrium systems, $x = 1$, so that one has the relation $TM(t, t_w) = C(t, t) - C(t, t_w)$. More generally, in non-equilibrium systems, a parametric plot $M(t, t_w)$ *vs.* $C(t, t) - C(t, t_w)$ is independent of $t_w$, and allows a direct determination of $x(C)$.

In the pre-asymptotic regime, the parametric plot of $M(t, t_w)$ *vs.* $C(t, t_w)$ (with $t$ as the parameter) will generally depend on $t_w$. Interesting information can nevertheless be extracted from this plot, as will be seen in the next section. In particular, a constant slope is indicative of a constant value of $X$, and a zero slope (plateau in $M$) corresponds to a loss of memory in the response.

A second property of the FDR is that under certain assumptions [7] it happens to coincide with the static Parisi function $x(q) = \int_0^q \mathrm{d}q' \, P(q')$, where $P(q)$ is the probability distribution of overlaps between real replicas of the same system. For the ferromagnetic case, $P(q)$ is trivial and $P(q) = \delta(q - M^2)$, where $M = M(T)$ is the magnetization. Therefore, we expect the FDR to be 1 if $1 > C > M^2$, and 0 if $M^2 > C$.

## 3 Simulation of a spinodal decomposition

The Monte-Carlo studies of reference [6] agreed qualitatively with the above behavior of the FDR, but to our knowledge no quantitative results are available yet. It is moreover clear that the asymptotic regime where the parametric plot Response/Correlation is supposed to collapse into a master-curve was not reached, the plot still conserving a dependence on $t_w$. In this work, we will be interested in a quantitative study of this pre-asymptotic behavior.

For this purpose, the stochastic partial differential equation (3) was numerically solved, in order to model a spinodal decomposition. Both time and space were discretized. A $1024 \times 1024$ square lattice with periodic boundary conditions was used. Spatial derivatives are treated using an implicit spectral method. Time derivatives were approximated using a simple Euler scheme. No real improvements have been obtained using a second or fourth order stochastic Runge-Kutta algorithms. The following recurrence relation is then obtained in the discretized Fourier space:

$$\phi(\mathbf{k}, t_{n+1}) = \left[ \frac{1}{1 + (k^4 - k^2)\Delta t} \right]$$
$$\times \left[ \phi(\mathbf{k}, t_n) - k^2 \Delta t (\phi^3(\mathbf{k}, t_n) - h(\mathbf{k})) + \sqrt{k}\eta \right]. \quad (9)$$

After discretization, the noise term $\eta$ is characterized by $\langle \eta(\mathbf{r_i}, t_n)\eta(\mathbf{r_j}, t_m) \rangle = 2T\Delta t/(\Delta x \Delta y)\delta_{ij}\delta_{nm}$. The algorithm is the following. Knowing the fields $\phi(\mathbf{r}, t_n)$ and $\phi^3(\mathbf{r}, t_n)$, the Fourier transforms $\phi(\mathbf{k}, t_n)$ and $\phi^3(\mathbf{k}, t_n)$ are computed. The recurrence relation (9) is then used to obtain $\phi(\mathbf{k}, t_{n+1})$. Fourier transforming again gives $\phi(\mathbf{r}, t_{n+1})$.

The influence of the parameters $\Delta x$, $\Delta y$, $T$ and $\Delta t$ on the numerical integration is discussed in the literature [11]. We chose $\Delta x = \Delta y = 0.5$, in order to get mesh-size independent results. The thickness of the domain walls in the late stage of the phase separation is indeed about $\xi = 1/\sqrt{2}$, where $\xi$ is the correlation length of the model (1). Their structure is hence sufficiently well described by the above discretization.
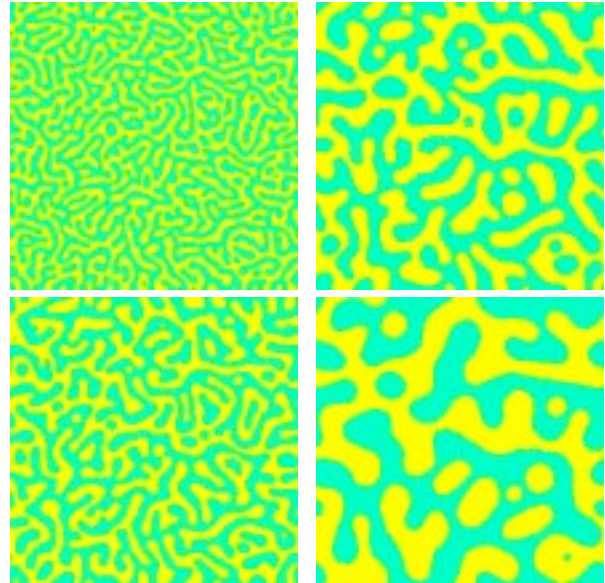


**Fig. 1.** Field configurations during the simulated phase separation for times 317, 1262, 5024, 20100. Each color represents one phase.

The role of $\Delta t$ is made less crucial by our choice of an implicit algorithm. The linear stability analysis of our algorithm gives indeed the following results: the "tangential bifurcation" [11], that is the small $k$ instability, is still present, but it is of course physically essential. On the contrary, the "subharmonic bifurcation" [11] does not exist any more. Hence, the only restriction on the time step is the average magnitude of the noise which has to be kept small in order to avoid numerical divergences. We chose then the highest possible value of $\Delta t$ compatible with the temperature $T$. A small temperature allows a large $\Delta t$, but obliges to work with a very small magnetic field when computing the response function (see below). We chose finally $T = 0.1$ and $\Delta t = 0.2$ in order to explore a large time range. For very short times, a spurious behavior (see the caption of Fig. 4) related to this rather large value of $\Delta t$ can be observed. We checked however that this deviations vanish when $\Delta t$ is smaller, and do not affect the long time evolution we are interested in.

Figure 1 presents different field configurations during the coarsening process. Looking at these pictures, it is clear that the coarsening process can be characterized by the typical size $L(t)$ of the domains. Growth laws are well known [10] and are $L(t) \sim t^{1/3}$ in the conserved case, and $L(t) \sim t^{1/2}$ in the non-conserved case. The preceding remark has a very interesting consequence which is known as *the scaling hypothesis*. As $L(t)$ is the only physically relevant length scale, statistical properties of the system are the same if we scale all the lengths by the factor $L(t)$ [10].

As in experiments, the measure of the domain size is obtained by computing the structure factor

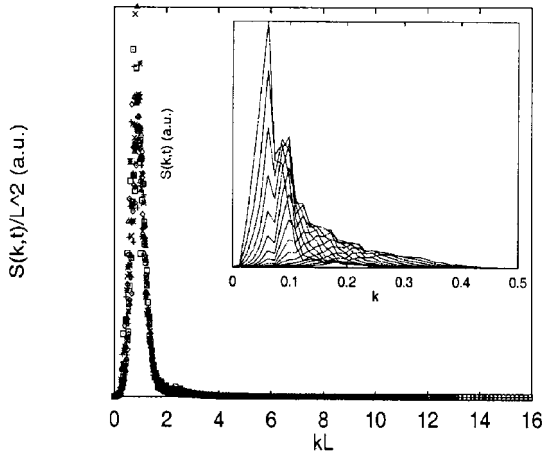$$S(\mathbf{k}, t) \equiv \langle \phi(\mathbf{k}, t)\phi(-\mathbf{k}, t) \rangle. \quad (10)$$

**Fig. 2.** Inset: circularly averaged structure factor for 15 different times from 317 to 23318. The largest time corresponds to the highest maximum. Main picture: the 15 curves collapse on a single scaling function.
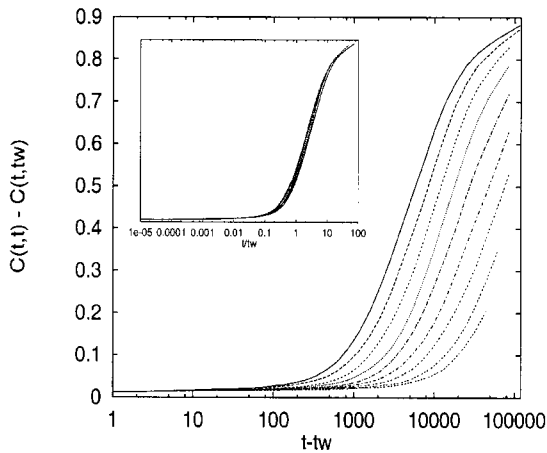


**Fig. 3.** Correlation function for waiting times (from left to right) $t_\mathrm{w} = 317, 502, 796, 1262, 2000, 3170, 5024, 7962$ and $12619$. Inset: the same curves as a function of $t/t_\mathrm{w}$.

The scaling hypothesis implies that it can be written as $S(\mathbf{k}, t) = L(t)^d g(kL(t))$, where $g$ is a scaling function. A convenient way of obtaining the growth law is to perform a circular average of $S(\mathbf{k}, t)$ and to compute then $\langle k \rangle \equiv \int \mathrm{d}k S(k, t)k / \int \mathrm{d}k S(k, t)$, which scales as $1/L(t)$. As in reference [11], we obtained the growth law $L(t) \sim t^{1/3}$, which is valid after a short transient period. The time evolution of the circularly averaged structure factor is depicted in the inset of Figure 2.

It has a clear maximum, corresponding to the wavevector $2\pi/L(t)$. This maximum shifts towards the small $k$, while its amplitude grows with time. The scaling hypothesis is verified plotting $S(k, t)/L(t)^2$ *vs.* $kL(t)$, all the curves collapsing on a very well defined scaling function $g$.

The autocorrelation functions for various waiting times $t_\mathrm{w}$ are shown in Figure 3. More precisely, the quantity $C(t, t) - C(t, t_\mathrm{w})$ is computed, in order to avoid normaliza-
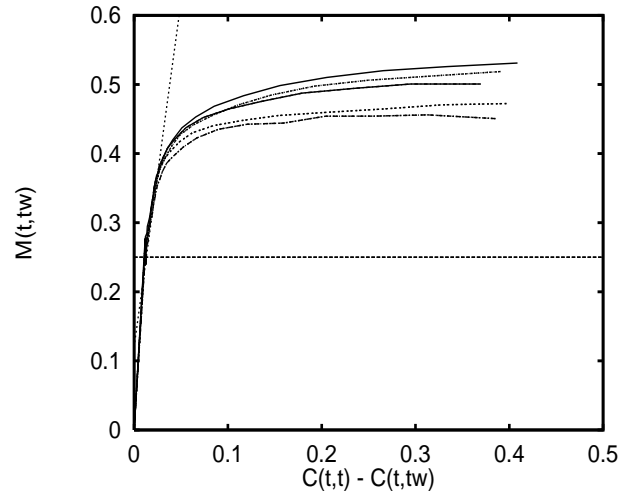


**Fig. 4.** Test of the fluctuation dissipation theorem. Waiting times are $t_\mathrm{w} = 317, 502, 796, 2000$ and $5024$. The horizontal line is the equilibrium value of the response and the dashed line is the FDT regime with slope $1/T$. This line has an intercept which is slightly positive: this is a time discretization effect, which can be made to vanish by reducing $\Delta t$.

tion problems of the correlation functions. As we are working with a soft spin system rather than a more usual Ising model, $C(t, t)$ is indeed a slowly varying function. More precisely $C(t, t) \sim C_\mathrm{eq} - L_0/L(t)$, where $C_\mathrm{eq}$ is the equilibrium correlation function in a bulk system and $L_0/L(t)$, is proportional to the interface density.

Two regimes may be distinguished: for times $t \ll t_\mathrm{w}$, the correlation is time translation invariant (TTI), and for times $t > t_\mathrm{w}$, aging is evident, with the TTI breakdown, and the correlation falls to 0. This scenario has been called *weak ergodicity breaking* [4]. The fluctuation dissipation theorem holds in the former, but is violated in the latter regime of times.

It is useful to use this behavior to introduce

$$C(t, t_\mathrm{w}) = C_\mathrm{st}(t - t_\mathrm{w}) + C_\mathrm{ag}(t, t_\mathrm{w}), \qquad (11)$$

where $C_\mathrm{st}$ and $C_\mathrm{ag}$ describe respectively a stationary and an aging part in the correlation.

The scaling hypothesis may be used to predict a scaling form for the aging part of the correlation function [10]:

$$C_\mathrm{ag}(t, t_\mathrm{w}) = f\left(\frac{L(t)}{L(t_\mathrm{w})}\right) = f\left(\frac{t}{t_\mathrm{w}}\right), \qquad (12)$$

$f$ being a scaling function. Equation (12) retains an explicit dependence on both times $t$ and $t_\mathrm{w}$, typical of an aging system. Such a scaling in the correlation function is called *simple aging*. As shown in the inset of Figure 3, this scaling form describes our results extremely well.

In order to complete the study of the fluctuation dissipation theorem, we have to compute the response function $M(t, t_\mathrm{w})$ to the static random field applied between $t_\mathrm{w}$ and $t$ (recall Sect. 2). The field amplitude has to be small to obtain a linear response. The best numerical test we found

for this purpose is the comparison of the time evolution of $\langle k \rangle$ with and without the magnetic field. When the field is present, the domain walls may be slowed down and even pinned if the field is too strong, so that the coarsening process is perturbed. This test is very sensitive, and we worked with small field amplitudes (between $h_0 = 0.035$ and 0.09) to ensure that the coarsening process was not affected.

With the correlation and the response functions, the parametric plot of $M(t, t_w)$ *vs.* $C(t, t_w)$ may be built. The data are shown in Figure 4, for various values of $t_w$, $t$ being the parameter. The curves are averaged over 14 to 32 realizations of the magnetic field. They are qualitatively the same as in the previous Monte-Carlo simulations [6], with a first part in which the FDT holds, corresponding to the times $t \ll t_w$. In a second part, which corresponds to times $t > t_w$, the FDT is obviously violated, with $M$ having a quasi-horizontal plateau. As discussed above, this plateau indicates the loss of long term memory in the response ($X = 0$ at long times), consistent with previous expectations.

The equilibrium value of the response function has been numerically computed by performing a similar simulation in an homogeneous system. This value is indicated in Figure 4 by an horizontal line (Note that this line can also be determined analytically, as discussed in the next section). From the arguments presented in the introduction, it could be expected that the long time plateau of the integrated response corresponds to that of a single domain. It is clearly seen from the data in Figure 4, however, that the approach to this asymptotic value is extremely slow.

As the extra response (*i.e.* the difference between the plateau value of $M(t, t_w)$ and the bulk response) can be attributed to the domain wall response, it is tempting to try to relate this response to the domain wall size. Between waiting times $t_w = 317$ and $t_w = 5024$, the size of the domains increases by a factor $(5024/317)^{1/3} \sim 2.51$, while the extra response is divided by only 1.36. The preasymptotic behavior of the FDR seems then to be related to $L(t_w)$ only through a non-trivial relation, which we explicitly discuss in the following section.

# 4 Analytical study of the fluctuation dissipation ratio

In order to study the fluctuation dissipation theorem, we have to compute separately the correlation and the response functions. Equation (12) will be sufficient for the present discussion, as we only need a scaling form for these functions. The response function may also be split into

$$M(t, t_w) = M_{eq}(t - t_w) + M_{ag}(t, t_w), \qquad (13)$$

exactly as we did for the correlation function.

We compute first $M_{eq} \equiv \lim_{t-t_w \to \infty} M_{eq}(t-t_w)$, which is in fact the static equilibrium response function of a single domain. It may be evaluated exactly at $T = 0$
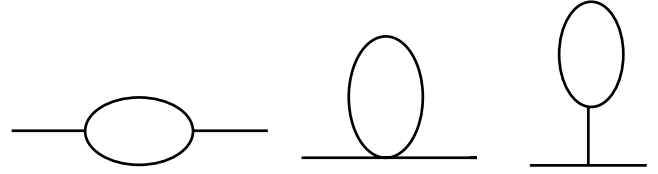


**Fig. 5.** The three Feynman graphs representing the "one loop" (proportional to the temperature $T$) corrections to equation (14).

(within the Gaussian approximation) and corrected perturbatively in powers of $T$. One easily finds for $T = 0$

$$M_{eq} = \int d^d k \frac{1}{k^2 + 1/\xi^2} \cdot \qquad (14)$$

Recall that we have $\xi = 1/\sqrt{2}$. In the simulation, the space is discretized, and this integral becomes then a discrete sum over the first Brillouin zone. A numerical evaluation of this sum yields a result in perfect agreement with the simulation result obtained for an homogeneous system, as described in the previous section. The first temperature corrections to equation (14) can be computed exactly (see Fig. 5), and are indeed found to be negligible at the temperatures we used.

Note that the integral (14) is divergent in the continuous theory: one has to introduce a spatial cutoff $a$, simulating the underlying lattice spacing. A convenient way of doing this is to multiply the integrand by $\exp(-k^2 a^2)$. Thus, the equilibrium response function scales with the cutoff as $M_{eq} \sim a^{2-d}$ for $d > 2$, and as $M_{eq} \sim \ln(a/\xi)$ for $d = 2$.

Next, we compute the "aging part" of the response function, which involves the response of the domain walls. This can be done using one of the "approximate theories for scaling functions" [10], which attempt to give an analytical expression for the scaling function $g$ of the structure factor, or equivalently for its Fourier transform. The spirit of these theories is to replace the field $\phi(\mathbf{r}, t)$, which at the late times of the coarsening process is $\pm 1$ outside the domain walls, by an auxiliary field $m(\mathbf{r}, t)$ varying smoothly in space. This allows to derive evolution equations for $m$ that may – with further approximations – become tractable. This method has already been used to justify analytically the scaling form (12) of the correlation function [12], and will be used below for the response function.

Unfortunately, such schemes have not been successfully developed for the conserved case. Our strategy will be then to study the non-conserved case within these approximations, and to give physical arguments to extend the validity of our result to the conserved case. For this purpose, we generalize a calculation of Bray in reference [13] of the response to an uniform magnetic field, to the staggered response to a random field. Following Bray, the equation for the auxiliary field $m$ reads (at $T = 0$)

$$\frac{\partial m}{\partial t} = \nabla^2 m - n_a n_b \nabla_a \nabla_b m + h|\nabla m|. \qquad (15)$$

The field $h$ depends now on space. Further simplifications, the validity of which we do not discuss here [10,13], are the replacements $n_a n_b \to \delta_{ab}/d$ (circular average), and $|\nabla m| \to \langle (\nabla m)^2 \rangle^{1/2}$. This computation scheme is near in spirit of the theory of Ohta, Jasnow and Kawasaki [14]. With these two assumptions, equation (15) becomes

$$\frac{\partial m}{\partial t} = D\nabla^2 m + a(t)h, \tag{16}$$

with $D = (d-1)/d$ and $a(t) = \langle (\nabla m)^2 \rangle^{1/2}$. The solution for $m$ is:

$$m(\mathbf{k}, t) = m(\mathbf{k}, 0)e^{-k^2 D t} + h(\mathbf{k}) \int_{t_{\mathrm{w}}}^t \mathrm{d}t'\, a(t')e^{-k^2 D(t-t')}. \tag{17}$$

Random initial conditions are conveniently chosen from a Gaussian distribution with mean zero and variance $\Delta$. The quantity we want to compute is the staggered magnetization $M_{\mathrm{ag}}(t, t_{\mathrm{w}}) = \langle h(\mathbf{r})\phi(\mathbf{r}, t)\rangle/{h_0}^2$. At late times of the coarsening process, the usual approximation $\phi \sim \mathrm{sgn}(m)$ can be made. Using the fact that the fields $m(\mathbf{k}, 0)$ and $h(\mathbf{k})$ are Gaussian, it is straightforward to obtain the staggered magnetization in term of the field $m$ and one gets:

$$M_{\mathrm{ag}}(t, t_{\mathrm{w}}) = \sqrt{\frac{2}{\pi}} \frac{\langle h(\mathbf{r})m(\mathbf{r})\rangle}{{h_0}^2 \sqrt{\langle m^2 \rangle}}. \tag{18}$$

Using equation (17), the relations $\langle m^2 \rangle \sim \Delta/(Dt)^{d/2}$ and $a(t) \sim \sqrt{\Delta}/t^{(d+2)/4}$ are also obtained and finally:

$$M_{\mathrm{ag}}(t, t_{\mathrm{w}}) \sim \int \mathrm{d}^d k\, M(k, t, t_{\mathrm{w}}), \tag{19}$$

$$M(k, t, t_{\mathrm{w}}) = \int_{t_{\mathrm{w}}}^t \mathrm{d}t'\, e^{-k^2 D(t-t')} \frac{(Dt)^{d/4}}{(Dt')^{(d+2)/4}}. \tag{20}$$

This integral over $k$ is divergent for large $k$. As in the equilibrium case we introduce a cutoff length $a$ via a term $\exp(-k^2 a^2)$. The integrals can now be performed and yield

$$\frac{M_{\mathrm{ag}}(t, t_{\mathrm{w}})}{M_{\mathrm{eq}}} \sim \frac{1}{t_{\mathrm{w}}^{1/2}} F\left(\frac{t}{t_{\mathrm{w}}}\right) \qquad d > 2,$$

$$\sim \frac{\ln(t_{\mathrm{w}}^{1/2}/a)}{\ln(a/\xi)t_{\mathrm{w}}^{1/2}} F\left(\frac{t}{t_{\mathrm{w}}}\right) \quad d = 2. \tag{21}$$

The scaling function $F$ is given by

$$F(\lambda) \equiv \lim_{A \to 0} \int_1^{\lambda} \frac{\mathrm{d}\lambda'}{A^{2-d}} \frac{\lambda^{d/4}}{\lambda'^{(d+2)/4}} \frac{1}{(\lambda - \lambda' + A^2)^{d/2}}. \tag{22}$$

Except in dimension $d = 2$, the cutoff $a$ disappears if the non-equilibrium response is measured in terms of the equilibrium one.

The meaning of this result can be better understood by considering the response associated to each spatial length scale separately. Defining $M(k, t, t_{\mathrm{w}})$ as the response to a

sinusoidal perturbation with wave-vector $k$, we can distinguish between two cases. For a wavelength larger than the domain size, $k \ll 1/L(t_{\mathrm{w}})$, we obtain

$$M(k, t, t_{\mathrm{w}}) \sim t^{d/4} \int_{t_{\mathrm{w}}}^t \frac{\mathrm{d}t'}{t'^{(d+2)/4}} \sim L(t_{\mathrm{w}})G\left(\frac{t}{t_{\mathrm{w}}}\right). \tag{23}$$

This is precisely equation (114) in reference [13]. The response of long wavelength modes grows with time, although their effect becomes negligible because the number of modes with $k \ll 1/L(t_{\mathrm{w}})$ decreases with time. On the other hand, for short wavelengths, $k \gg 1/L(t_{\mathrm{w}})$, the integral can be approximated to find

$$M(k, t, t_{\mathrm{w}}) \sim \frac{1}{k^2} \frac{1}{t^{1/2}}, \tag{24}$$

which is in fact a very simple result. The susceptibility of an elastic surface (a flat domain wall) when a field with wave-vector $\mathbf{k}$ is applied is proportional to $1/k^2$, and the density of interfaces is proportional to $1/L(t) \sim 1/t^{1/2}$.

The behavior of the aging part of the response function may now be simply evaluated as the sum of two terms. The first one is the contribution of small wave-vectors $k \ll 1/L(t_{\mathrm{w}})$. We have already shown that for the non-conserved case, it became negligible as $L \to \infty$. We can safely assume that this is a general statement, as the influence of the long wavelengths is even smaller in the conserved case (*cf.* the Cahn-Hilliard equation). The second one, corresponding to wave-vectors $k \gg 1/L(t_{\mathrm{w}})$ scales as $\int_{1/L}^{1/a} \mathrm{d}^d k/k^2 L$. The long-time response

$$\frac{M_{\mathrm{ag}}(t, t_{\mathrm{w}})}{M_{\mathrm{eq}}} \sim \frac{1}{L(t_{\mathrm{w}})} F\left(\frac{L(t)}{L(t_{\mathrm{w}})}\right) \qquad d > 2,$$

$$\sim \frac{\ln(L(t_{\mathrm{w}})/a)}{\ln(a/\xi)L(t_{\mathrm{w}})} F\left(\frac{L(t)}{L(t_{\mathrm{w}})}\right) \quad d = 2, \tag{25}$$

is entirely dominated by the short wavelengths. For a non-conserved dynamics, with $L \sim t^{1/2}$, equation (21) is retrieved.

The result (25) is now in a form *independent of the dynamics* (conserved or non-conserved order parameter) of the system. Hence, for any coarsening system, equations (12) and (25) give an analytical evaluation of the response associated to the domain walls in the aging part. As expected, this response vanishes at long times, so that in the asymptotic regime $t, t_{\mathrm{w}} \to \infty$, the value $x(C) = 0$ is obtained in all dimensions.

The scaling (25) is tested in Figure 6, where we plot $t_{\mathrm{w}}^{1/3} M_{\mathrm{ag}}$ *vs.* $\ln(t_{\mathrm{w}})$. Here $M_{\mathrm{ag}}$ is defined as the difference between the plateau value obtained in the simulation and the equilibrium response. Our data are obviously consistent with the above assumption. Hence we conclude that the extra response obtained in the simulation actually corresponds to the domain wall response, and will asymptotically vanish. This vanishing, however, is extremely slow, so that we can hardly expect to see it in any numerical simulation.
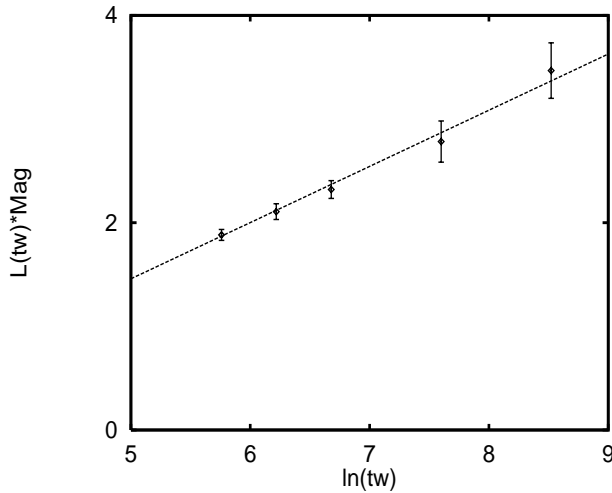
**Fig. 6.** Test of the scaling (25) which predicts a linear dependence of $L(t_{\mathrm{w}})M_{\mathrm{ag}}$ w.r.t. $\ln(t_{\mathrm{w}})$. The dashed line fits this dependence very well.

## 5 Conclusion

As mentioned above, the important measurable (dynamical) difference between mean-field glass models and coarsening models is the presence – or absence – of a long term memory in the response functions. In view of the relations this has in certain cases with the Parisi function, the question as to whether real glasses have a value of $X$ that stays different from, or tends slowly to zero is sometimes taken as the modern version of the old "droplet" *versus* "mean-field" debate of the 80's. Experimentally, however, the difference between $X$ tending logarithmically to zero or staying constant might not be very dramatic. What is important, however, is that if one has a law for the integrated response of the form

$$M_{\mathrm{ag}}(t, t_{\mathrm{w}}) \sim A(t_{\mathrm{w}})F\left(\frac{L(t)}{L(t_{\mathrm{w}})}\right), \qquad (26)$$

then if $A(t) \sim (L(t))^{-1}$ a large aging response is necessarily linked to very slow scaling laws $L(t)$. The fact that we have found for $d = 2$ a relation $A(t) \sim \ln(L(t))/L(t)$ shows that indeed it is possible for the response to fall slower than the inverse of the rate of growth $L$, and one can have relatively large long term memories together with rather fast growth laws.

Another important conclusion of the present study, which confirms earlier numerical work, is that the domain walls can have a large contribution to the response in the pre-asymptotic regime, but almost exclusively given by their deformation on relatively short lengths. This elastic contribution can be considered thermalized, and its contribution makes longer the segment of slope $1/T$ in the parametric plot of the integrated response *versus* correlation. Apart from that, the plot is flat for smaller values of $C$. This is still very different from mean-field glass models, in which the out of equilibrium contribution is "thermalized" at an effective temperature different from $T$.

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